Markov Chains

Toolbox

- Search: uninformed/heuristic
- Adversarial search
- Probability
- Bayes nets
  - Naive Bayes classifiers
- Statistical inference

Reasoning over time

- In a Bayes net, each random variable (node) takes on one specific value.
  – Good for modeling static situations.

- What if we need to model a situation that is changing over time?

Example: Comcast

- In 2004 and 2007, Comcast had the worst customer satisfaction rating of any company or gov't agency, including the IRS.
- I have cable internet service from Comcast, and sometimes my router goes down. If the router is online, it will be online the next day with prob=0.8. If it's offline, it will be offline the next day with prob=0.4.
- How do we model the probability that my router will be online/offline tomorrow? In 2 days?

Example: Waiting in line

- You go to the Apple Store to buy the latest iPhone. Every minute, the first person in line is served with prob=0.5.
- Every minute, a new person joins the line with probability

  1 if the line length=0

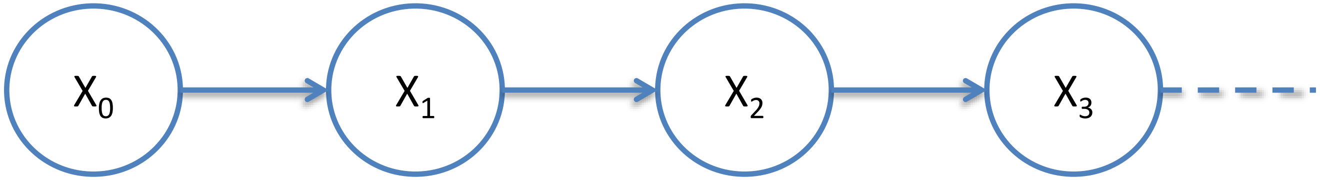
  2/3 if the line length=1

  1/3 if the line length=2

  0 if the line length=3
- How do we model what the line will look like in 1 minute? In 5 minutes?

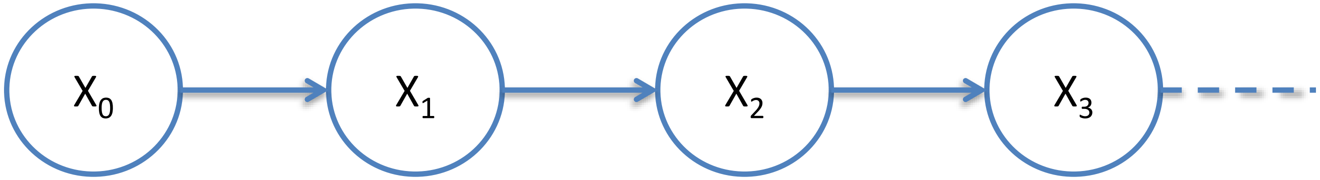
Markov Chains

- A Markov chain is a type of Bayes net with a potentially infinite number of variables (nodes).
- Each variable describes the state of the system at a given point in time (t).



Markov Chains

- Markov property:
  $P(X_t \mid X_{t-1}, X_{t-2}, X_{t-3}, \ldots) = P(X_t \mid X_{t-1})$
- Probabilities for each transition are identical:
  $P(X_t \mid X_{t-1}) = P(X_1 \mid X_0)$



Markov Chains

- Since these are just Bayes nets, we can use standard Bayes net ideas.
  - Shortcut notation: $X_{i:j}$ will refer to all variables X_i through X_j, inclusive.
- Common questions:
  - What is the probability of a specific event happening in the future?
  - What is the probability of a specific sequence of events happening in the future?

An alternate formulation

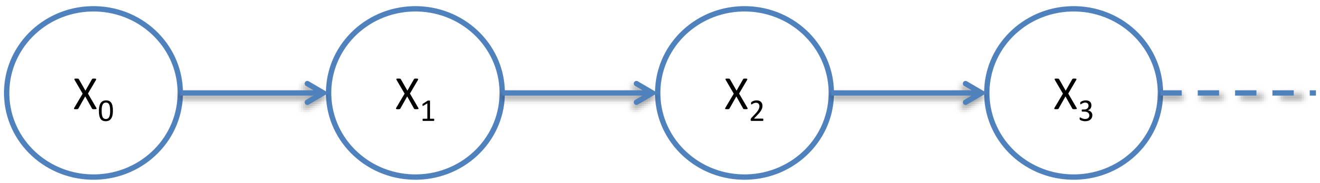
- We have a set of states, S.
- The Markov chain is always in *exactly one* state at any given time t.
- The chain transitions to a new state at each time t+1 based only on the current state at time t.

$$p_{ij} = P(X_{t+1} = j \mid X_t = i)$$

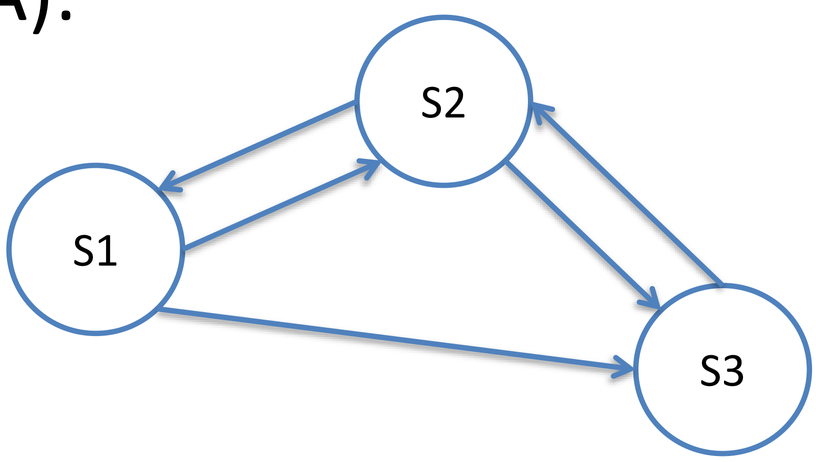
- Chain must specify p_{ij} for all i and j, and starting probabilities for $P(X_0 = j)$ for all j.

Two different representations

- As a Bayes net:



- As a state transition diagram (similar to a DFA/NFA):



Formulate Comcast in both ways

- I have cable internet service from Comcast, and sometimes my router goes down. If the router is online, it will be online the next day with prob=0.8. If it's offline, it will be offline the next day with prob=0.4.
- Let’s draw this situation in both ways.
- Assume on day 0, probability of router being down is 0.5.

Comcast as a BN

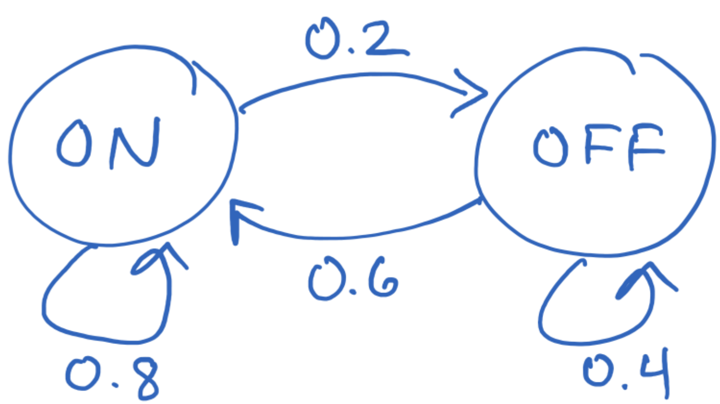
$P(X_0=on)=\frac{1}{2}$

$P(X_0=off)=\frac{1}{2}$

$P(X_1 \mid X_0)$

X_0	$X_1=on$
on	0.8
off	0.6

Comcast as a state diagram



Comcast

- What is the probability my router is offline for 3 days in a row (days 0, 1, and 2)?
  - $P(X_2=off, X_1=off, X_0=off)$?
  - $P(X_2=off|X_0=off, X_1=off)*P(X_0=off, X_1=off)$ *[mult rule]*
  - $P(X_2=off|X_0=off, X_1=off)*P(X_1=off|X_0=off)*P(X_0=off)$
  - $P(X_2=off|X_1=off)*P(X_1=off|X_0=off)*P(X_0=off)$
  - $p_{off,off} * p_{off,off} * P(X_0=off)$

$$P(x_{0:t}) = P(x_0) \prod_{i=1}^{t} P(x_i \mid x_{i-1})$$

More Comcast

- Suppose I don't know if my router is online right now (day 0). What is the prob it is offline tomorrow?
  - $P(X_1=\text{off})$
  - $P(X_1=\text{off}) = P(X_1=\text{off}, X_0=\text{on}) + P(X_1=\text{off}, X_0=\text{off})$
  - $P(X_1=\text{off}) = P(X_1=\text{off} | X_0=\text{on}) * P(X_0=\text{on})$
    $+ P(X_1=\text{off} | X_0=\text{off}) * P(X_0=\text{off})$

$$P(X_{t+1}) = \sum_{x_t} P(X_{t+1} \mid x_t) P(x_t)$$

More Comcast

- Suppose I don't know if my router is online right now (day 0). What is the prob it is offline **the day after tomorrow**?
  - $P(X_2=\text{off})$
  - $P(X_2=\text{off}) = P(X_2=\text{off}, X_1=\text{on}) + P(X_2=\text{off}, X_1=\text{off})$
  - $P(X_2=\text{off}) = P(X_2=\text{off} | X_1=\text{on}) * P(X_1=\text{on})$

    $+ P(X_2=\text{off} | X_1=\text{off}) * P(X_1=\text{off})$

$$P(X_{t+1}) = \sum_{x_t} P(X_{t+1} \mid x_t) P(x_t)$$

Markov chains with matrices

- Define a transition matrix for the chain:

$$T = \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}$$

- Each row of the matrix represents the transition probabilities **leaving** a state; each column represents the **next** state.
- Let v_t = a row vector representing the probability that the chain is in each state at time t.
- $v_t = v_{t-1} * T$

Formulate this matrix

- If the stock market is up one day, then it will be up the next day with prob=0.7.
- If it's down one day, it will be down the next day with prob=0.4.

Mini-forward algorithm

- Suppose we are given the value of X_t or a probability distribution over X_t and we want to predict X_{t+1}, X_{t+2}, X_{t+3}…
- Make row vector $v_t = P(X_t)$
  - Note that v_t can be something like [1, 0] if you know the true value of X_t, or it can be a distribution over values.
- $v_{t+1} = v_t * T$
- $v_{t+2} = v_{t+1} * T = v_t * T * T = v_t * T^2$
- $v_{t+3} = v_t * T^3$
- $v_{t+n} = v_t * T^n$

Back to the Apple Store...

- You go to the Apple Store to buy the latest iPhone.
- Every minute, a new person joins the line with probability

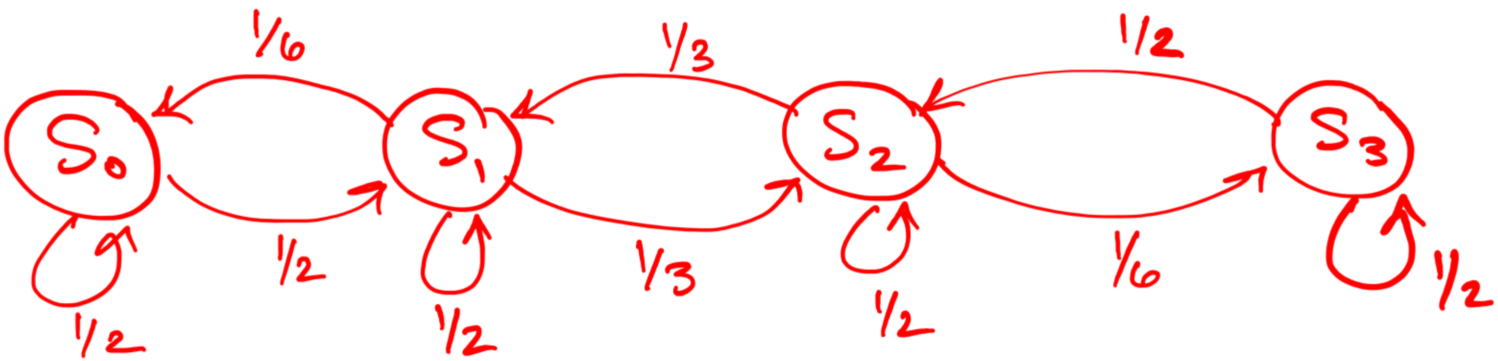
  1 if the line length=0

  2/3 if the line length=1

  1/3 if the line length=2

  0 if the line length=3
- Immediately after (in the same minute), the first person is helped with prob = 0.5
- Model this as a Markov chain, assuming the line starts empty. Draw the state transition diagram.
- What is T? What is v_0?

- Every minute, a new person joins the line with probability
  - 1 if the line length=0
  - 2/3 if the line length=1
  - 1/3 if the line length=2
  - 0 if the line length=3
- Immediately after (in the same minute), the first person is helped with prob = 0.5.



S_0 (0 ppl in line)

+1 new person

$S_0 \to S_1$

$S_0 \to S_0$

S_1 (1 person in line)

+1 ($p=\frac{2}{3}$) | +0 ($p=1/3$)

($p=\frac{1}{2}$) −1 → S_1 ($\frac{1}{3}$); −0 → S_2 ($\frac{1}{3}$); −1 → S_0 ($\frac{1}{6}$); −0 → S_1 ($\frac{1}{6}$)

S_3 (3 ppl)

$$T = \begin{array}{c|cccc} & S_0 & S_1 & S_2 & S_3 \\ \hline S_0 & 1/2 & 1/2 & 0 & 0 \\ S_1 & 1/6 & 1/2 & 1/3 & 0 \\ S_2 & 0 & 1/3 & 1/2 & 1/6 \\ S_3 & 0 & 0 & 1/2 & 1/2 \end{array}$$

$$V_0 = [1 \quad 0 \quad 0 \quad 0]$$

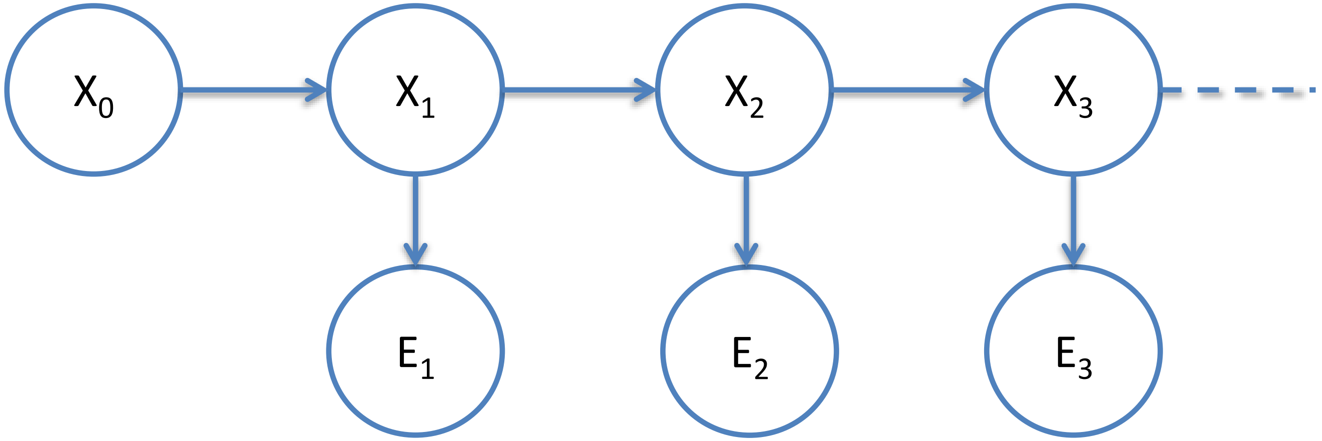
- Markov chains are pretty easy!
- But sometimes they aren't realistic…

- What if we can't directly know the states of the model, but we can see some indirect evidence resulting from the states?

Weather

- Regular Markov chain
  - Each day the weather is rainy or sunny.
  - $P(X_t = \text{rain} \mid X_{t-1} = \text{rain}) = 0.7$
  - $P(X_t = \text{sunny} \mid X_{t-1} = \text{sunny}) = 0.9$
- Twist:
  - Suppose you work in an office with no windows. All you can observe is weather your colleague brings their umbrella to work.

Hidden Markov Models

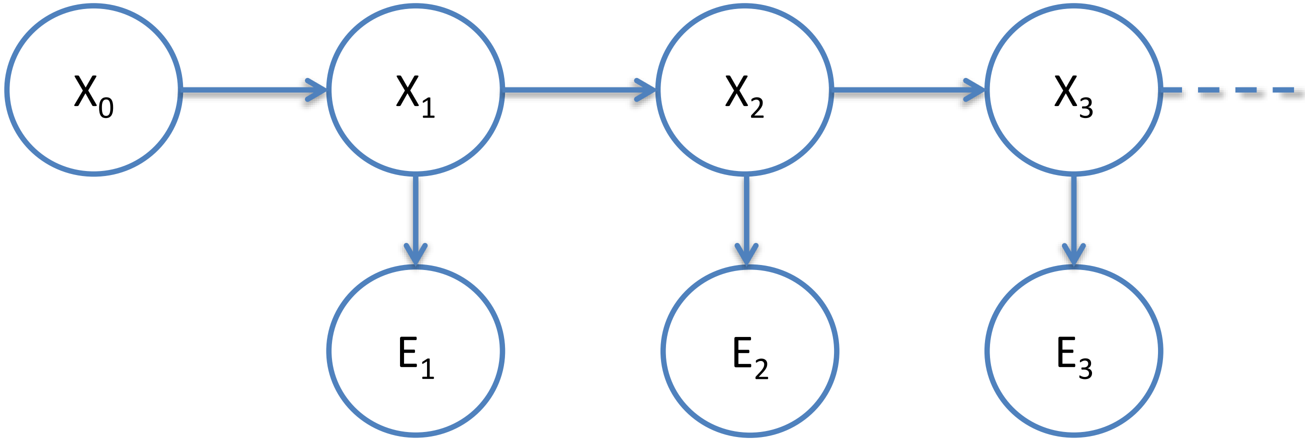


- The X's are the state variables (never directly observed).
- The E's are evidence variables.

Common real-world uses

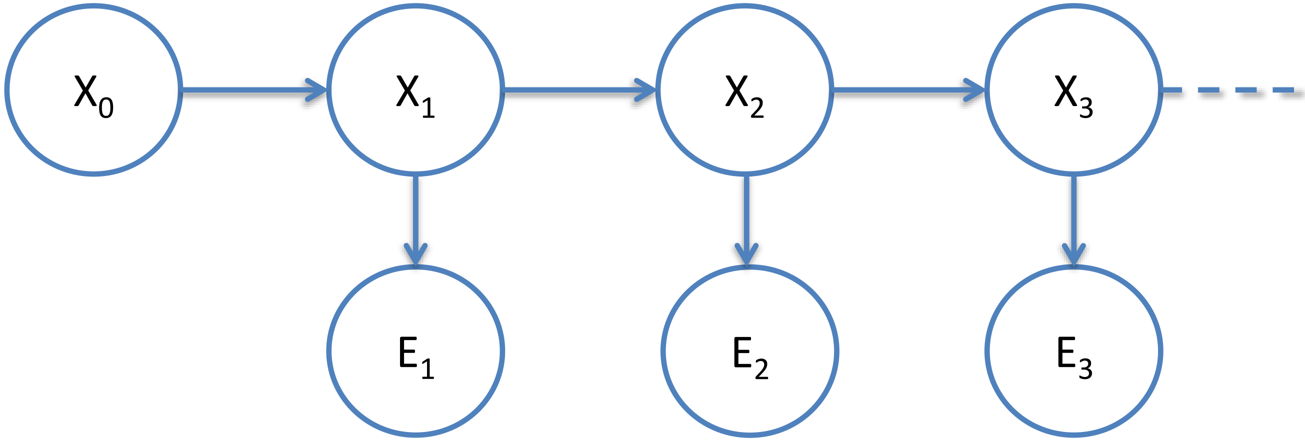
- Speech processing:
  - Observations are sounds, states are words or phonemes.
- Localization:
  - Observations are inputs from video cameras or microphones, state is the actual location.
- Video processing (example):
  - Extracting a human walking from each video frame. Observations are the frames, states are the positions of the legs.

Hidden Markov Models



- $P(X_t \mid X_{t-1}, X_{t-2}, X_{t-3}, \ldots) = P(X_t \mid X_{t-1})$
- $P(X_t \mid X_{t-1}) = P(X_1 \mid X_0)$
- $P(E_t \mid X_{0:t}, E_{0:t-1}) = P(E_t \mid X_t)$
- $P(E_t \mid X_t) = P(E_1 \mid X_1)$

Hidden Markov Models



- What is $P(X_{0:t}, E_{1:t})$?

$$P(X_0) \prod_{i=1}^{t} P(X_i \mid X_{i-1}) P(E_i \mid X_i)$$

Common questions

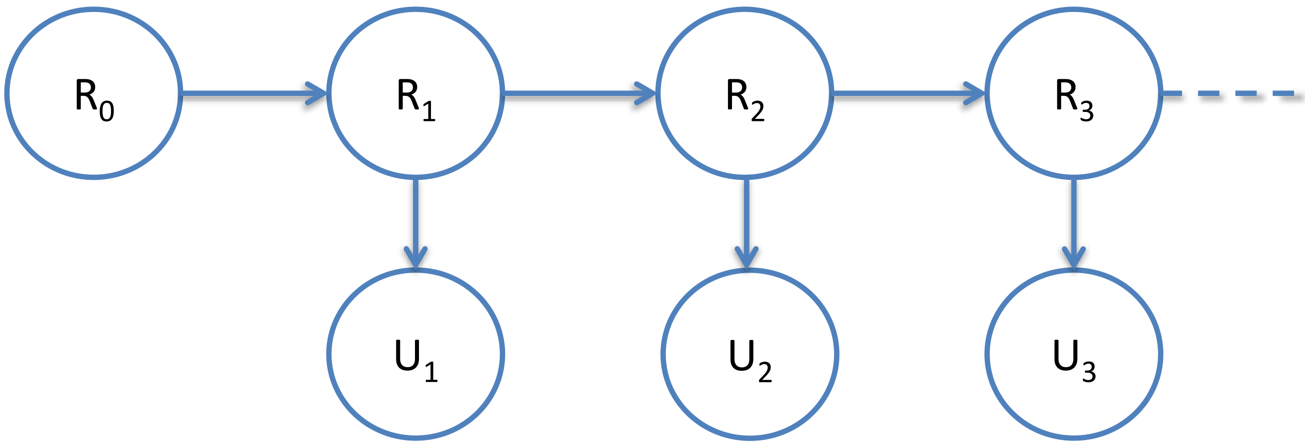
- **Filtering**: Given a sequence of observations, what is the most probable *current* state?
  - Compute $P(X_t \mid e_{1:t})$
- **Prediction**: Given a sequence of observations, what is the most probable *future* state?
  - Compute $P(X_{t+k} \mid e_{1:t})$ for some $k > 0$
- **Smoothing**: Given a sequence of observations, what is the most probable *past* state?
  - Compute $P(X_k \mid e_{1:t})$ for some $k < t$

Common questions

- **Most likely explanation:** Given a sequence of observations, what is the most probable sequence of states?
  – Compute $\underset{x_{1:t}}{\operatorname{argmax}} P(x_{1:t} \mid e_{1:t})$

- **Learning**: How can we estimate the transition and sensor models from real-world data? (Future machine learning class?)

Hidden Markov Models



- $P(R_t = yes \mid R_{t-1} = yes) = 0.7$
  $P(R_t = yes \mid R_{t-1} = no) = 0.1$
- $P(U_t = yes \mid R_t = yes) = 0.9$
  $P(U_t = yes \mid R_t = no) = 0.2$

Filtering

- Filtering is concerned with finding the most probable "current" state from a sequence of evidence.
- Let's compute this.

Recall the "mini-forward algorithm"

For Markov chains:

$$P(X_{t+1}) = \sum_{x_t} P(X_{t+1} \mid x_t) P(x_t)$$

with matrices: $v_{t+1} = v_t * T$, with $v_0 = P(X_0)$

For HMM's:

$$P(X_{t+1} \mid e_{1:t+1}) =$$

$$\alpha P(e_{t+1} \mid X_{t+1}) \sum_{x_t} P(X_{t+1} \mid x_t) P(x_t \mid e_{1:t})$$

Forward algorithm

- Today is Day 2, and I've been pulling all-nighters for two days!
- My colleague brought their umbrella on days 1 and 2.
- What is the probability it is raining today?
  - that is, find $P(X_t \mid e_{1:t})$ [*filtering*]
- Assume initial distribution of rain/not-rain for Day 0 is 50-50.

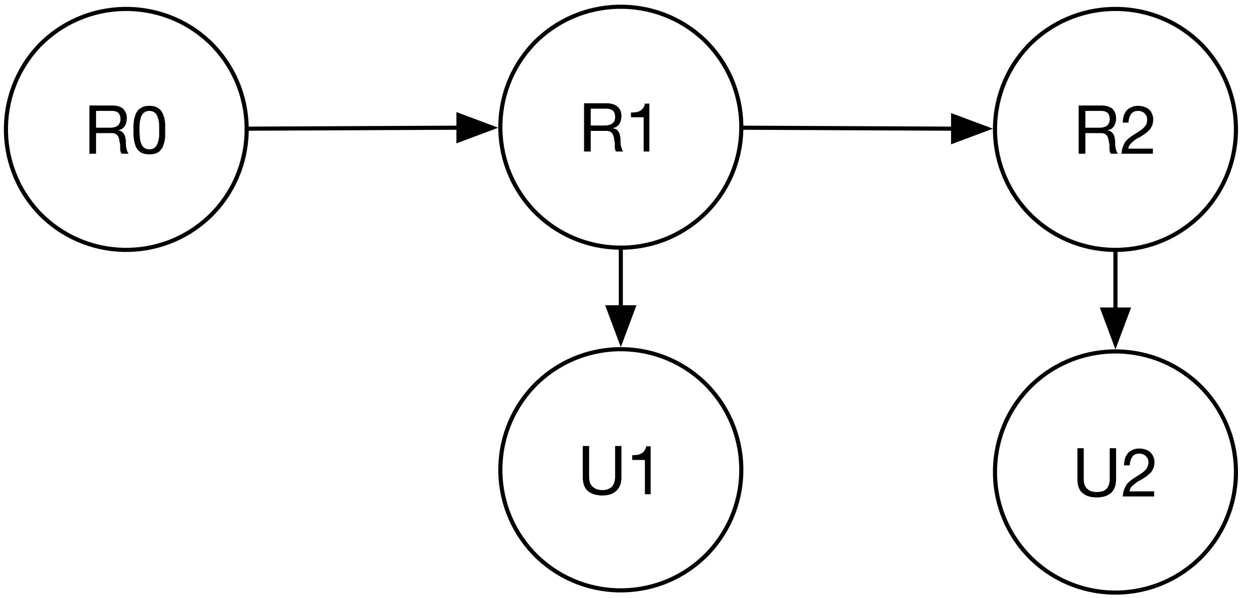
Matrices to the rescue!

- Define a transition matrix T as normal.
- Define a sequence of observation matrices O_1 through O_t.
- Each O matrix is a diagonal matrix with the entries corresponding to observation at time t given each state.

$$f_{1:t+1} = \alpha f_{1:t} \cdot T \cdot O_{t+1}$$

where each f is a row vector containing the probability distribution at timestep t.

f1:0=[0.5, 0.5]   f1:1=[0.75, 0.25]   f1:2=[0.846, 0.154]




```
T  = [0.7, 0.3]
     [0.1, 0.9]

O1 = [0.9, 0.0]
     [0.0, 0.2]

O2 = [0.9, 0.0]
     [0.0, 0.2]
```

f1:0 = P(R0) = [0.5, 0.5]

f1:1 = P(R1 | u1) = $\boldsymbol{\alpha}$ * f1:0 * T * O1 = $\boldsymbol{\alpha}$[0.36, 0.12] = [0.75, 0.25]

f1:2 = P(R2 | u1, u2) = $\boldsymbol{\alpha}$ * f1:1 * T * O2 = $\boldsymbol{\alpha}$[0.495, 0.09] = [.846, .154]

Forward algorithm

- Note that the forward algorithm only gives you the probability of X_t taking into account evidence at times 1 through t.
- In other words, say you calculate $P(X_1 \mid e_1)$ using the forward algorithm, then you calculate $P(X_2 \mid e_1, e_2)$.
  - Knowing e2 changes your calculation of X1.
  - That is, $P(X_1 \mid e_1) \mathrel{!=} P(X_1 \mid e_1, e_2)$

Backward algorithm

- Updates previous probabilities to take into account new evidence.
- Calculates $P(X_k \mid e_{1:t})$ for $k < t$
  - aka **smoothing**. (not the same kind of smoothing as in Naïve bayes)

Backward algorithm

- Algorithm generates a *backward vector* b for every timestep t.
  - This vector is based on the observation at time k and the *next day's* backward vector.

    $$b_{k:t} = T \cdot O_k \cdot b_{k+1:t}$$

  - The initial backwards vector is for day t+1 and is a column vector of all 1's.

    $$b_{t+1:t} = [1; \cdots ; 1]$$

Backwards algorithm

- Each backward vector is used to *scale* the previous day's forward vector.
- After normalization, this is the updated probability for day k.

$$P(X_k \mid e_{1:t}) = \alpha f_{1:k} \times b_{k+1:t}$$

- (Remember, that multiplication above is an item by item multiplication, not a matrix multiplication.)

Backward matrices

- Main equations:

$$b_{k:t} = T \cdot O_k \cdot b_{k+1:t}$$

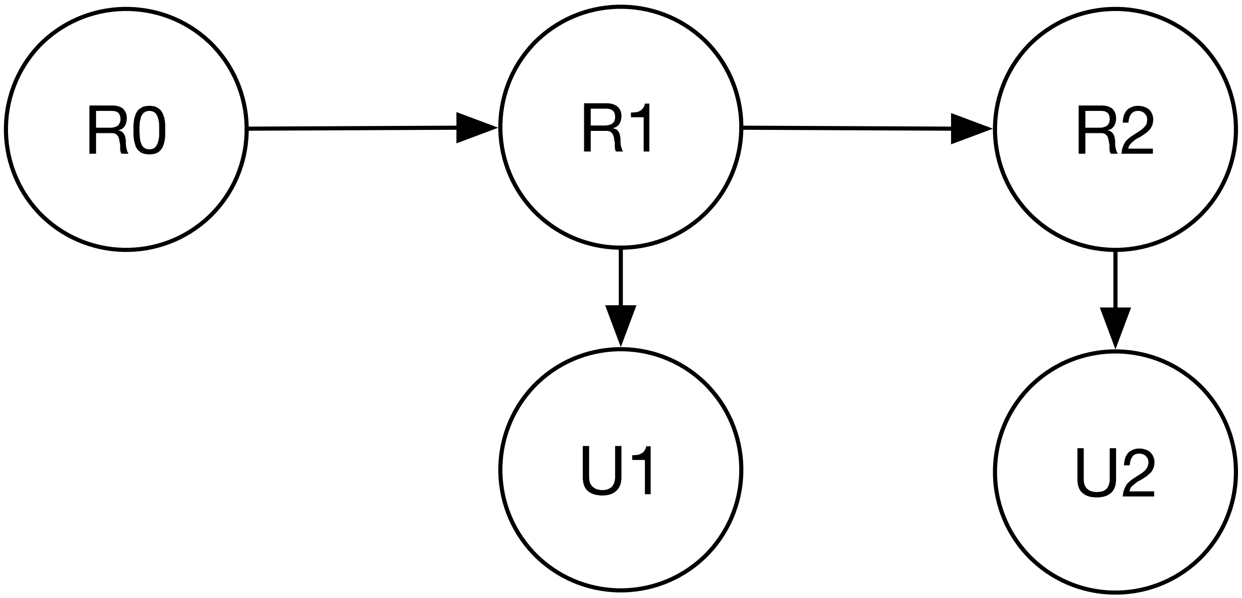
$$b_{t+1:t} = [1; \cdots ; 1]$$ (column vector of 1s)

$$P(X_k \mid e_{1:t}) = \alpha f_{1:k} \times b_{k+1:t}$$

f1:0=[0.5, 0.5] f1:1=[0.75, 0.25] f1:2=[0.846, 0.154]

b1:2=[0.4509, 0.1107] b2:2=[0.69, 0.27] b3:2=[1; 1]

mult=[0.803, 0.197] mult=[0.885, 0.115]




```
T  = [0.7, 0.3]
     [0.1, 0.9]

O1 = [0.9, 0.0]
     [0.0, 0.2]

O2 = [0.9, 0.0]
     [0.0, 0.2]
```

b3:2 = [1; 1]

b2:2 = T * O2 * b3:2 = [0.69, 0.27]

P(R1 | u1, u2) = α f1:1 x b2:2 = α[0.5175, 0.0675] = [0.885, 0.115]

b1:2 = T * O1 * b2:2 = [0.4509, 0.1107]

P(R0 | u1, u2) = α f1:0 x b1:2 = α[0.2255, 0.0554] = [0.803, 0.197]

Forward-backward algorithm

$$f_{1:0} = P(X_0)$$

$$f_{1:t+1} = \alpha f_{1:t} \cdot T \cdot O_{t+1}$$

Compute these forward from X_0 to wherever you want to stop (X_t)

$$b_{t+1:t} = [1; \cdots ; 1]$$

$$b_{k:t} = T \cdot O_k \cdot b_{k+1:t}$$

$$P(X_k \mid e_{1:t}) = \alpha f_{1:k} \times b_{k+1:t}$$

Compute these backwards from X_{t+1} to X_0.